TD 1 solutions

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Mathematical formalism.

Problem 1. *Prove that, for every odd integer $x \in \mathbb{Z}$, there exists some integer $y \in \mathbb{Z}$ such that $x^2 = 8y + 1$.*

Solution 1. We could do a proof by induction (where we assume that the statement holds true for $x = 2k + 1$ and show that it then holds true for $x = 2k + 3$) but, for no real reason, we opt instead for a direct proof here.

Write $x = 2k + 1$ for some $k \in \mathbb{Z}$, so that

$$x^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1.$$

It then suffices to show that $k(k + 1)$ is even (i.e. divisible by 2), but this follows from the pigeonhole principle: of any n consecutive integers, at least one must divide by n. Said differently, if k is odd, then $k + 1$ is even, and so their product is even; if instead k is even then ($k + 1$ is odd, and) the product is again even. □

For the following two problems it can be simpler to use the following equivalent definition of injectivity: a function $f: A \to B$ is injective if (and only if) it admits a left inverse, i.e. there exists some function $g: B \to A$ such that $g \circ f = \mathrm{id}_A$.

Problem 2. *Let $c \neq 0$ be a real number. Show that the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = cx$ is injective.*

Solution 2.[1] Let $x, y \in \mathbb{R}$ be such that $f(x) = f(y)$, i.e. $cx = cy$. But since $c \neq 0$, we can divide by c to obtain that $x = y$. □

Problem 3. *Let $f, g: \mathbb{R} \to \mathbb{R}$ be injective. Show that their composition $g \circ f: \mathbb{R} \to \mathbb{R}$ is also injective.*

[1] Alternatively, we could show that the function $f^{-1,l}: \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x/c$ is a left inverse of f.

Solution 3.[2] Let $x, y \in \mathbb{R}$ be such that $(g \circ f)(x) = (g \circ f)(y)$, i.e. $g(f(x)) = g(f(y))$. Then, since g is injective, $f(x) = f(y)$; since f is injective, $x = y$. □

Problem 4. *Fix a real number $x \neq 1$. Show,* by induction*, that, for every non-negative integer $n \in \mathbb{N}$, the following equality holds.*

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

Solution 4.[3] Note that the equality is indeed true for $n = 0$, so assume that it also holds true for some $n \in \mathbb{N}$. Then

$$\begin{aligned} 1 + x + x^2 + \ldots + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\ &= \frac{x^{n+1} - 1}{x - 1} + \frac{x^{n+1}(x - 1)}{x - 1} \\ &= \frac{x^{n+1} - 1}{x - 1} + \frac{x^{n+2} - x^{n+1}}{x - 1} \\ &= \frac{x^{n+2} - 1}{x - 1}. \end{aligned}$$ □

Dedekind cuts.

The following lemma can prove useful in some of these exercises.

Lemma. *Any Dedekind cut D is bounded above, that is, there exists some $B \in \mathbb{Q}$ such that $d < B$ for all $d \in D$.*

Proof. *Assume that D is not bounded above, and let $q \in \mathbb{Q}$ be arbitrary. Then there exists some $d \in D$ such that $q < d$ (otherwise q would be an upper bound for D). By the downwards-closed property of Dedekind cuts, $q \in D$, whence $\mathbb{Q} \subseteq D$, contradicting the properness of D.* □

Problem 5. *Let x and y be Dedekind cuts, and define their* sum *$x \oplus y$ as*

$$x \oplus y = \{d + e \mid d \in x \text{ and } e \in y\}.$$

Prove that $x \oplus y$ is a Dedekind cut.

[2] Again, using the 'left-inverse' definition of injectivity, we could simply show that the composition admits a left inverse, given exactly by the composition of the left inverse of g and the left inverse of f

[3] The reason that we ask specifically for a proof by induction is because a direct proof is pretty quick: simply multiply the left-hand side by $(x - 1)$ and note that every term except -1 and x^{n+1} cancels.

Solution 5. First note that $x \oplus y$ is non-empty, since both x and y are non-empty. Then, since both x and y are bounded above (say, by B and C, respectively), $x \oplus y$ is also bounded above (by $B + C$), and so in particular cannot contain any rational number greater than this bound (e.g. $B + C + 1$). This shows that $x \oplus y$ is a proper subset of $\mathbb{Q}$.

Now let $q \in \mathbb{Q}$ be such that $q < d+e$ for some $d+e \in x \oplus y$. Then, setting $\varepsilon = d+e-q > 0$, we have that $q + \varepsilon = d + e$, whence[4] $q = d + e - \varepsilon$. But $e - \varepsilon < e$ and so is an element of y, and d is an element of x by definition, so their sum is an element of $x \oplus y$, i.e. $q \in x \oplus y$. This shows that $x \oplus y$ is downwards closed.

Finally, let $d + e \in x \oplus y$. Then, since x has no maximum, there exists some $d' \in x$ with $d < d'$ (and similarly for $e' < e$), and so we have $d + e < d' + e' \in x \oplus y$. This shows that $x \oplus y$ has no maximal element, and we are done. □

Problem 6. *Define $Z = \{q \in \mathbb{Q} \mid q < 0\}$ (where $0 \in \mathbb{Q}$), which we assume is a Dedekind cut. Show that, for any Dedekind cut x, we have that $x \oplus Z = x$.*

Solution 6. First, let $d + q \in x \oplus Z$. By definition, $q < 0$, and so $d + q < d$. Then, by the fact that x is downwards closed, $d + q \in x$. So $x \oplus Z \subseteq x$.

Now let $d \in x$. Since x has no maximum[5], there exists some $d' \in x$ such that $d < d'$. Then $0 > d - d' \in Z$, and $d = d' + (d - d')$ is an expression for d as a sum of an element of x (namely d') and an element of Z (namely $d - d'$). Thus $x \subseteq x \oplus Z$, and so $x = x \oplus Z$.

Problem 7. *Define the product $x \odot y$ of two* non-negative *Dedekind cuts x and y by*

$$x \odot y = Z \cup \{d \cdot e \mid d \in x \setminus Z \text{ and } e \in y \setminus Z\}.$$

Show that $x \odot y$ is a Dedekind cut.

Solution 7. Note that if either one of x or y is equal to Z then $x \cdot y = Z$, which we already know to be a Dedekind cut, so we can assume that both x and y have at least one non-negative element.

Since Z is non-empty so too is the union of Z with any other set, and hence $x \odot y$ is not empty. Similarly to Problem 5, we can use the fact that both x and y are bounded above (by B and C, respectively, say): $x \odot y$ is bounded above by $B \cdot C$ (since every element of Z is less than zero). This gives us the properness of $x \cdot y$.

Now let $q \in \mathbb{Q}$ be such that $q < f$ for some $f \in x \cdot y$. There are two possible cases: either $f \in Z$, or $f = d \cdot e$ for some $d \in x$, $e \in y$. In the former case, $q \in Z$, since Z is downwards closed, and so $q \in x \cdot y$. In the latter case, let $m = (d \cdot e)/q \in \mathbb{Q}$, so that $q \cdot m = d \cdot e$. Then $q = d \cdot (e/m)$, where $d \in x$ by definition and $e/m \in y$ since $e/m < e$ and y is downwards

[4] In the TDs I split this up as $q = (d - \varepsilon/2) + (e - \varepsilon/2)$, which is not necessary, but is maybe more aesthetically pleasing to you. If so, then use it.

[5] A (sometimes useful) equivalent way of saying the fact that x has no maximum is that, for all $d \in x$, there exists some $\varepsilon > 0$ (in $\mathbb{Q}$!) such that $d + \varepsilon \in x$.

closed (N.B. we implicitly use the fact that everything here is non-negative). This shows us that $x \cdot y$ is downwards closed.

Finally, let $f \in x \cdot y$, and again consider the two cases: either $f \in Z$, or $f = d \cdot e$ for some $d \in x$, $e \in y$. So in the former case it suffices to find some $f' \in x \cdot y$ that is non-negative, since $f < 0$, but this is easy: take f' to be the product of any non-negative element of x with any non-negative element of y (recalling our assumption at the start of this proof). In the latter case, by the non-maximal property of x and y, we can find some $d' \in x$ and $e' \in y$ such that $d < d'$ and $e < e'$. Then $d \cdot e < d' \cdot e' \in x \cdot y$, and so $x \odot y$ has no maximum. Thus $x \odot y$ is a Dedekind cut. $\square$

Problem 8. *For a Dedekind cut x, we define its* negative $\ominus x$ *by*

$$\ominus x = \{-q \mid q \in \mathbb{Q} \setminus x \text{ and } q \text{ is not a minimum element of } \mathbb{Q} \setminus x\}.$$

(a) (Warm up.) *Show that $\ominus Z = Z$.*

(b) (Useful technical step.) *Show that $\mathbb{Q} \setminus x$ is closed* upwards*, i.e. if $q \in \mathbb{Q} \setminus x$ and $q' \in \mathbb{Q}$ is such that $q' > q$ then $q' \in \mathbb{Q} \setminus x$.*

(c) *Show that $\ominus x$ is indeed a Dedekind cut.*

(d) (Another technical step.) *Show that, for every $r < 0$, there exists some $d \in x$ such that $d - r \in \mathbb{Q} \setminus x$.*

(e) *Show that $x \oplus (\ominus x) = Z$.*

Solution 8.

(a) By definition, $\mathbb{Q} \setminus Z = \{q \in \mathbb{Q} \mid q \geqslant 0\}$, so

$$\begin{aligned} \ominus Z &= \{-q \mid q \in \mathbb{Q} \setminus Z \text{ and } q \text{ is not a minimum element of } \mathbb{Q} \setminus Z\} \\ &= \{-q \in \mathbb{Q} \mid q \geqslant 0 \text{ and } q \text{ is not a minimum element of } \mathbb{Q} \setminus Z\} \\ &= \{-q \in \mathbb{Q} \mid q \geqslant 0\} \setminus \{0\} \\ &= \{-q \in \mathbb{Q} \mid q > 0\} \\ &= \{q \in \mathbb{Q} \mid q < 0\} = Z. \end{aligned}$$ $\square$

(b) Let $q \in \mathbb{Q} \setminus x$ and $q' \in \mathbb{Q}$ be such that $q' > q$. If it were the case that $q' \notin \mathbb{Q} \setminus x$ (i.e. $q' \in x$) then, by the downwards-closed property of x, we would have that $q \in x$, since $q < q'$. But this would be a contradiction, so it must be the case that $q' \in \mathbb{Q} \setminus x$. $\square$

(c) We know that $\ominus x$ is not the whole of $\mathbb{Q}$ since x is not $\varnothing$; we know that $\ominus x$ is not $\varnothing$ since $\mathbb{Q} \setminus x$ contains infinitely many elements (because e.g. x is bounded above) and so removing at most one element (the minimum, if it exists) will leave us with at least infinitely many elements, and infinity is more than zero.

Since $\mathbb{Q} \setminus x$ is upwards closed, $\ominus x$ is downwards closed (since it contains exactly[6] the negatives of elements of $\mathbb{Q} \setminus x$).[7]

Finally, $\ominus x$ contains no maximal element by definition, since the minimal element of $\mathbb{Q} \setminus x$ is exactly the element that would become the maximum after reversing the sign of every element. Thus $\ominus x$ is a Dedekind cut. ☐

(d) [8] Let $d + e \in x \oplus (\ominus x)$, so that, in particular, $-e \in \mathbb{Q} \setminus x$, i.e. $-e \notin x$. By the same argument as that in (b) we know that $-e$ must be an upper bound for x, hence, in particular, $d < -e$, whence $0 > d + e \in Z$. So $x \oplus (\ominus x) \subseteq Z$.

Now let $q \in Z$, so that $q < 0$. What we would like to do is to write $q = d - (d - q)$ for some $d \in x$ and $-(d - q) \in \ominus x$, but this would require showing that there exists some such d *and* that $d - q$ is not a minimal element of $\mathbb{Q} \setminus x$. If, however, we can do this, then we are done. With the language of supremums, this would be much easier, but we must make do with the tools we have, so we proceed as follows.

First of all, note that, if such a $d \in x$ exists, then we can always take some $d' \in x$ with $d < d'$, whence $d - q < d' - q \in \mathbb{Q} \setminus x$ (by (c)) and $q = d' - (d' - q)$. This means that, if $d - q$ *were* the minimal element of $\mathbb{Q} \setminus x$, we wouldn't have a problem, because we can instead work with a strictly greater element.

Now, for actually proving that such a $d \in x$ *does* exist, we work by contradiction: assume that, for *all*[9] $d \in x$, we have that $d - q \in x$. We claim that this implies that $d - nq \in x$ for any $n \in \mathbb{N}$, which we can prove by induction.[10] Now let $p \in \mathbb{Q}$ be arbitrary. If $p < d$ then $p \in x$ by the downwards-closed property. If $p \geqslant d$ then we write $p = \rho/\rho'$, $d = \delta/\delta'$, and $q = \kappa/\kappa'$, where all numerators and denominators are integers; all denominators positive; and $\kappa < 0$. Then

$$\frac{p-d}{-q} = \frac{\kappa'(\rho\delta' - \rho'\delta)}{-\kappa\rho'\delta'}$$

[6]Apart from maybe the minimum element.

[7]This could maybe be explained slightly better, but I think that this is enough detail for the proof. I hate the phrase 'it should be obvious if you think about it', but if I had to use it just once in these solutions, it would be here.

[8]This is by far the most fiddly and unenlightening proof that we have seen so far. Although it is an entirely valid proof, it does not make for fun reading, and it doesn't really help us to better understand *why* the statement is true. Unfortunately, you will sometimes come across such proofs, so at the very least this serves as a way of getting your feet wet in the ocean of confusing proofs that surrounds the island of real analysis.

[9]When taking the converse of a statement you exchange every $\exists$ with a $\forall$ (and vice versa) *except* for the last $\exists$, which you negate, e.g $\exists a\,\forall b\,\exists c\,\forall d\,\exists x$ becomes $\forall a\,\exists b\,\forall c\,\exists d\nexists x$.

[10]The case $n = 1$ is exactly our contradiction hypothesis. If it is true for some $n \in \mathbb{N}$, then $p - (n+1)r = (p - nr) - r \in x$, since $(p - nr) \in x$ (by our inductive hypothesis) and thus $(p - nr) - r \in x$ by our contradiction hypothesis.

and so we can define $n \in \mathbb{N}$ by

$$n = 1 + \frac{-\kappa\rho'\delta'(p-d)}{-q} > \frac{p-d}{-q}.$$

A tedious calculation shows that $p < d - nq$, with $d - nq \in x$ by the above claim (that we proved by induction). Since x is closed downwards, this means that, again, $p \in x$, and so $\mathbb{Q} \subseteq x$, which contradicts the properness of x. $\square$