TD 1

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Mathematical formalism.

The objective of these exercises is for you to practise writing proofs: the exercises themselves are not supposed to be difficult, but you should write the solutions as best as you can.

Problem 1. *Prove that, for every odd integer $x \in \mathbb{Z}$, there exists some integer $y \in \mathbb{Z}$ such that $x^2 = 8y + 1$.*

Problem 2. *Let $c \neq 0$ be a real number. Show that the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = cx$ is injective.*

Problem 3. *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be injective. Show that their composition $f \circ g \colon \mathbb{R} \to \mathbb{R}$ is also injective.*

Problem 4. *Fix a real number $x \neq 1$. Show,* by induction*, that, for every non-negative integer $n \in \mathbb{N}$, the following equality holds.*

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

Dedekind cuts.

In the following exercises, $+$, $\cdot$, $<$, and $\leqslant$ all refer to operations on the *rational numbers*, and so we cannot use their properties (as we know them) on the *irrational* numbers: all we can use is their properties on $\mathbb{Q}$, as well as the properties of Dedekind cuts.

Problem 5. *Let x and y be Dedekind cuts, and define their* sum *$x \oplus y$ as*

$$x \oplus y = \{d + e \mid d \in x \text{ and } e \in y\}.$$

Prove that $x \oplus y$ is a Dedekind cut.

Problem 6. *Define* $Z = \{q \in \mathbb{Q} \mid q < 0\}$ *(where* $0 \in \mathbb{Q}$*), which we assume is a Dedekind cut. Show that, for any Dedekind cut* x*, we have that* $x \oplus Z = x$*.*

Recall that, in the lectures, we defined a total order $\preceq$ on Dedekind cuts by

$$x \preceq y \iff x \subseteq y.$$

We say that a Dedekind cut x is *non-negative* if $Z \preceq x$.

Problem 7. *Define the product* $x \odot y$ *of two* non-negative *Dedekind cuts* x *and* y *by*

$$x \odot y = Z \cup \{d \cdot e \mid d \in x \setminus Z \text{ and } e \in y \setminus Z\}.$$

Show that $x \odot y$ *is a Dedekind cut.*

Problem 8. *For a Dedekind cut* x*, we define its* negative $\ominus x$ *by*

$$\ominus x = \{-q \mid q \in \mathbb{Q} \setminus x \text{ and } q \text{ is not a minimum element of } \mathbb{Q} \setminus x\}.$$

(a) (Warm up.) *Show that* $\ominus Z = Z$*.*

(b) (Useful technical step.) *Show that* $\mathbb{Q} \setminus x$ *is* closed upwards*, i.e. if* $q \in \mathbb{Q} \setminus x$ *and* $q' \in \mathbb{Q}$ *is such that* $q' > q$ *then* $q' \in \mathbb{Q} \setminus x$*.*

(c) *Show that* $\ominus x$ *is indeed a Dedekind cut.*

(d) (Another technical step.) *Show that, for every* $r < 0$*, there exists some* $d \in x$ *such that* $d - r \in \mathbb{Q} \setminus x$*.*

(e) *Show that* $x \oplus (\ominus x) = Z$*.*