TD 2 solutions

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Problem 1. *Let S be a non-empty subset of $\mathbb{R}$ and M an upper bound for S. Prove that $M = \sup S$ if and only if, for all $\varepsilon > 0$, there exists some $x \in S$ such that $M - \varepsilon \leqslant x \leqslant M$.*

Solution 1.[1] First, assume that $M = \sup S$, and let $\varepsilon > 0$ be arbitrary. Then assume, for a proof by contradiction, that there does *not* exist some $x \in S$ such that $x \in [M-\varepsilon, M]$. Then, for all $x \in S$, since $x \leqslant M$ (by definition of the supremum) and $x \notin [M-\varepsilon, M]$, we must have that $x < M - \varepsilon$. But then $M - \varepsilon$ is an upper bound for S that is strictly smaller than M (since $\varepsilon > 0$), which contradicts the fact that M is the supremum, whence our initial assumption (that there does *not* exist some $x \in S$ such that $x \in [M-\varepsilon, M]$) must be false.

Now assume that, for all $\varepsilon > 0$, there exists some $x_\varepsilon \in S$ such that $x_\varepsilon \in [M - \varepsilon, M]$. Recall that M is defined to be an upper bound of S, so it just remains to show that it is minimal among such bounds. For this, let N be another upper bound of S. If $N < M$ then $M - N > 0$, whence there exists some $x_\varepsilon \in S$ such that $N < x_\varepsilon$ (e.g. $\varepsilon = (M-N)/2 > 0$, since then $N = M - 2\varepsilon < M - \varepsilon$), which contradicts the assumption that N is an upper bound. Thus $N \geqslant M$, and M is a minimal upper bound for S. $\square$

Problem 2.[2] *We say that $\lim_{n\to\infty} a_n = +\infty$ if, for every real number $R \in \mathbb{R}$, there exists some integer $N \in \mathbb{N}$ such that $a_n > R$ for all $n \geqslant N$. Show that a divergent monotone-increasing sequence converges to infinity in this sense.*

Solution 2. Let $(a_n)_{n\in\mathbb{N}}$ be a divergent monotone-increasing sequence in $\mathbb{R}$. In particular then, $a_{n+1} \geqslant a_n$ for all $n \in \mathbb{N}$, since (a_n) is monotone-increasing. Further, for all $n \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ such that $m > n$ and $a_m > a_n$, since if not then our sequence would converge to a_n, contradicting our initial hypotheses.[3]

[1] This proof is slightly wordy and over-detailed, but this is arguably a better fault than being vague and imprecise. Think of this proof as a supremum of the set of proofs ordered by 'amount of detail necessary for a good proof'.

[2] This problem is sort of a converse to the *Bolzano–Weierstrass theorem* (a fundamental result in real analysis), and also related to (one of) the *monotone convergence theorem(s)*.

[3] This is a quick way of writing a proof by contradiction; much less cumbersome than 'Assume, for contradiction, that ... Then ... But this gives a contradiction.' But remember that brevity is not always the same as

For a contradiction, assume that our sequence (a_n) does *not* tend to infinity. This means that there exists some $R \in \mathbb{R}$ such that $a_n \leqslant R$ for all $n \in \mathbb{N}$, i.e. our sequence is bounded above. But then there exists some supremum, i.e. $M \in \mathbb{R}$ such that $M = \sup\{a_n \mid n \in \mathbb{N}\}$. Using Problem 1, we know that, for all $\varepsilon > 0$, there exists some $n(\varepsilon) \in \mathbb{N}$ such that $a_{n(\varepsilon)} \in [M-\varepsilon, M]$. However, since (a_n) is monotone increasing, we know that $a_m \geqslant a_{n(\varepsilon)}$ for all $m > n(\varepsilon)$, whence

$$\forall \varepsilon > 0 \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad \forall m > n \quad |a_m - M| < \varepsilon$$

by taking $n = n(\varepsilon/2)$, and this statement says exactly that the limit of (a_n) exists and is equal to M, which contradicts the assumption that (a_n) is divergent. ☐

Note that we cannot *use some argument along the lines of*

$$\lim_{n\to\infty} |a_{n+1} - a_n| = 0 \implies \lim a_n < +\infty$$

since this is not *true (consider $a_n = \ln n$, for example).*

Problem 3. *Let $x_n = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{\ldots+\sqrt{n}}}}}$ for $n \in \mathbb{N} \setminus \{0\}$.*

(a) *Show that $x_n < x_{n+1}$ for all $n \in \mathbb{N} \setminus \{0\}$.*

(b) *Show that $x_{n+1}^2 \leqslant 1 + \sqrt{2}x_n$ for all $n \in \mathbb{N} \setminus \{0\}$.*

(c) *Hence show that $(x_n)_{n\in\mathbb{N}\setminus\{0\}}$ is bounded above by 2. Deduce that its limit as $n \to \infty$ exists.*

Solution 3.

(a) *We give a complete proof, as well as some open-ended sketches of proofs. Note that we are studying the sequence $(x_n)_{n\geqslant 1}$, i.e. there is no x_0 term.*

(Recursion.) Since the derivative of $f(y) = \sqrt{y}$ (as a function $(1,\infty) \to \mathbb{R}$) is positive, we know that f is increasing, i.e. that $\sqrt{y+\varepsilon} > \sqrt{y}$ for any $\varepsilon > 0$, whence $\sqrt{y+a} > \sqrt{y+b}$ for any $a > b$. Thus $\sqrt{n+\sqrt{n+1}} > \sqrt{n}$. Thus $\sqrt{n-1+\sqrt{n+\sqrt{n+1}}} > \sqrt{n-1+\sqrt{n}}$. Thus .... Thus $x_{n+1} > x_n$.

(Strong induction.) Note that $x_1 = 1$ and $x_2 = \sqrt{1+\sqrt{2}} > \sqrt{1+1} > 1$, so the statement is definitely true for the first two terms. Now look assume (for a proof by *strong*[4] induction) that $x_{n+1} > x_n$ for all $n < m$ for some $m \in \mathbb{N}$. It then remains to be shown that $x_{m+1} > x_m$. *How could we do this?*

clarity: proofs should be written with as specific an audience in mind as possible, and then aimed towards that audience.

[4] If you are not familiar with strong induction (also known as *complete* induction) then look it up.

(Recursive functions.) Could we somehow define this sequence in terms of recursively-defined functions and then use another argument about the derivative being positive? Basically, could we write our recursive proof above in a 'neater' way?

(Taylor series.) We might be inclined to use the Taylor expansion

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$$

but this only converges for $|x| < 1$, and $\sqrt{2} > 1$, so this seems to be a dead end.[5] Could we still use a similar idea somehow?

(b) First we note that

$$\begin{aligned} x_{n+1}^2 &= 1 + \sqrt{2 + \sqrt{3 + \sqrt{\dots + \sqrt{n+1}}}} \\ &= 1 + \sqrt{2}\sqrt{1 + \frac{1}{2}\sqrt{3 + \sqrt{\dots + \sqrt{n+1}}}} \end{aligned}$$

and so it remains only to show that

$$\sqrt{1 + \frac{1}{2}\sqrt{3 + \sqrt{\dots + \sqrt{n+1}}}} \leqslant \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{\dots + \sqrt{n}}}}}.$$

But

$$\begin{aligned} \sqrt{1 + \frac{1}{2}\sqrt{3 + \sqrt{\dots + \sqrt{n+1}}}} &= \sqrt{1 + \frac{1}{\sqrt{4}}\sqrt{3 + \sqrt{\dots + \sqrt{n+1}}}} \\ &= \sqrt{1 + \sqrt{\frac{3}{2} + \sqrt{\frac{4}{4} + \sqrt{\frac{5}{8} + \sqrt{\dots + \frac{\sqrt{n+1}}{2^{n-1}}}}}}}. \end{aligned}$$

For $n \geqslant 2$ it is evident that $\sqrt{n+1} \leqslant 2^{n-1}n$, whence if we replace every integer i in the expression for x_n with $\frac{\sqrt{i+1}}{2^{i-1}}$ then we will have an expression that is less than or equal to x_n, but this is exactly the required inequality.

(c) Let's proceed by induction (since we know that $x_1 = 1 < 2$) and assume that $x_n < 2$ for some $n \in \mathbb{N}$. Then $x_{n+1}^2 \leqslant 1 + \sqrt{2}x_n < 1 + 2\sqrt{2} < 1 + 2\sqrt{\frac{9}{4}} = 2$.

[5] If you know some complex analysis, then why do we have this radius of convergence?

Problem 4.

(a) *Let* $(a_n)_{n\in\mathbb{N}}$ *be a bounded sequence. Define a sequence* $(b_n)_{n\in\mathbb{N}}$ *by*

$$b_n = \sup\{a_k \mid k \geqslant n\}.$$

Prove that (b_n) *converges. (This limit is called the* limit superior *of* (a_n)*, often written as* $\limsup$*.)*

(b) *Without redoing the proof of the previous part of this exercise, show that the* limit inferior *always converges as well, where* $\liminf a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geqslant n\}$.

The limit superior and limit inferior are very useful tools in real analysis, for many reasons[6]*, hence their inclusion in these problems. An equivalent definition of the limit superior (resp. limit inferior) is that it is the largest (resp. smallet) subsequential limit.*[7]

Solution 4.

(a) If we could show that the sequence $(b_n)_{n\in\mathbb{N}}$ is bounded and monotone then we would be done, by some form of monotone convergence theorem (c.f. the footnotes for Problem 2). But the sequence is indeed bounded *below* by $\sup_n a_n$ (which exists, since $(a_n)_{n\in\mathbb{N}}$ is bounded and thus, in particular, bounded above).[8] Further, $b_{n+1} \leqslant b_n$, by a similar argument:

$$\{k \in \mathbb{N} \mid k \geqslant n+1\} \subset \{k \in \mathbb{N} \mid k \geqslant n\}.$$

Thus $(b_n)_{n\in\mathbb{N}}$ converges.

(b) Note that $\inf T \geqslant \inf S$ for $T \subseteq S \subseteq \mathbb{R}$ and that $(a_n)_{n\in\mathbb{N}}$ is bounded and thus, in particular, bounded below, so that $\inf_n a_n$ exists, which provides an *upper* bound for $c_n = \inf\{a_k \mid k \geqslant n\}$.

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[6]For example, the limit of a sequence exists if and only if the limit inferior and the limit superior are equal, and in this case is equal to them; and in general we have that $\inf_n x_n \leqslant \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \leqslant \sup_n x_n$. They also satisfy *sub-additivity* (for $\limsup$), i.e.

$$\limsup_{n\to\infty}(a_n + b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty b_n}$$

whenever the right-hand side is well defined; and *super-additivity* (for $\liminf$), i.e.

$$\liminf_{n\to\infty}(a_n + b_n) \geqslant \liminf_{n\to\infty} a_n + \liminf_{n\to\infty b_n}$$

whenever the right-hand side is well defined.

[7]To show that this definition is equivalent to the one given in the TD sheet is not entirely trivial.

[8]This is a special case of the fact that $\sup T \leqslant \sup S$ for any subset $T \subseteq S \subseteq \mathbb{R}$, which follows by a definition chase. The idea is that all the elements of T are also elements of S, and so any upper bound for S is also be an upper bound for T. Alternatively, think of it like this: by throwing away the biggest elements of S we could get a smaller upper bound, but if we throw away the smallest elements of S then the upper bound will not change, so for any subset $T \subseteq S$, the only thing that can happen to the upper bound is that it gets smaller.