

TD 2

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Problem 1. Let S be a non-empty subset of \mathbb{R} and M an upper bound for S . Prove that $M = \sup S$ if and only if, for all $\varepsilon > 0$, there exists some $x \in S$ such that $M - \varepsilon \leq x \leq M$.

The following exercises are all from Section 2.5 of the book.

Problem 2. We say that $\lim_{n \rightarrow \infty} a_n = +\infty$ if, for every real number $R \in \mathbb{R}$, there exists some integer $N \in \mathbb{N}$ such that $a_n > R$ for all $n \geq N$. Show that a divergent monotone-increasing sequence converges to infinity in this sense.

Problem 3. Let $x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{\dots + \sqrt{n}}}}}$ for $n \in \mathbb{N} \setminus \{0\}$.

- (a) Show that $x_n < x_{n+1}$ for all $n \in \mathbb{N} \setminus \{0\}$.
- (b) Show that $x_{n+1}^2 \leq 1 + \sqrt{2}x_n$ for all $n \in \mathbb{N} \setminus \{0\}$.
- (c) Hence show that $(x_n)_{n \in \mathbb{N} \setminus \{0\}}$ is bounded above by 2. Deduce that its limit as $n \rightarrow \infty$ exists.

Problem 4.

- (a) Let $(a_n)_{n \in \mathbb{N}}$ be a bounded sequence. Define a sequence $(b_n)_{n \in \mathbb{N}}$ by

$$b_n = \sup\{a_k \mid k \geq n\}.$$

Prove that (b_n) converges. (This limit is called the limit superior of (a_n) , often written as \limsup .)

- (b) Without redoing the proof of the previous part of this exercise, show that the limit inferior always converges as well, where $\liminf a_n = \lim_{n \rightarrow \infty} \inf\{a_k \mid k \geq n\}$.