# TD 3 solutions

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**Problem 1. (2.6.A)**
*Show that* $(a_n) = \left(\frac{n\cos^n(n)}{\sqrt{n^2+2n}}\right)_{n=1}^{\infty}$ *has a convergent subsequence.*

For explicitly constructing the subsequence, I'm simply going to refer you guys to Example 2.6.6 (p. 53) in the book, since that describes exactly the sort of method that you will use to solve this exercise. I will, however, describe some heuristic sketch of an idea: it would be lovely if we could find some sequence $(I_n)$ of natural numbers such that $\lim_{n\to\infty} I_n \pmod{2\pi} \to \pi/2$, since then $\cos^{I_n}(I_n)$ would tend to zero 'very quickly', thanks to the fact that, for $x < 1$, $x^n < \ldots < x^2 < x < 1$.

The 'real' proof, however, is just to show that this sequence is bounded and then apply Bolzano-Weierstrass. To prove boundedness, note that

$$\begin{aligned}\left|\frac{n\cos^n(n)}{\sqrt{n^2+2n}}\right| &\leqslant \left|\frac{n}{\sqrt{n^2+2n}}\right| \times |\cos^n(n)| \\ &\leqslant \left|\frac{n}{\sqrt{n^2+2n}}\right| \times 1 \\ &= \sqrt{\frac{n^2}{n^2+2n}} \\ &\leqslant \sqrt{1} = 1.\end{aligned}$$

**Problem 2. (2.6.B)**
*Does the sequence* $(b_n) = \left(n + \cos(n\pi)\sqrt{n^2+1}\right)_{n=1}^{\infty}$ *have a convergent subsequence?*

Note that there are only two values that $\cos(n\pi)$ can take: $\{\cos(n\pi) \mid n \in \mathbb{N}\} = \{-1, 1\}$, so let's consider a 'construction by exhaustion'.

i) If we pick some sequence $(n_m) \subset \mathbb{N}$ (i.e. in order to specify some subsequence $(b_{n_m})$ of $(b_n)$) such that $\cos(n_m\pi) = 1$ then the subsequence $(b_{n_m})$ will tend to $+\infty$, i.e. it won't converge in $\mathbb{R}$.

ii) If we pick some sequence $(n_m) \subset \mathbb{N}$ such that $\cos(n_m\pi)$ changes sign infinitely often (i.e., for all $m \in \mathbb{N}$ there exists some $M > m$ such that $\cos n_m = -\cos n_M$) then $|b_{n_m} - b_{n_M}| \geqslant 2\sqrt{m^2+1} \to \infty$, which would contradict convergence of the subsequence, and so, if a subsequence is to converge, it must at least be such that the $\cos(n_m\pi)$ term is eventually[1] always $-1$.

iii) Finally then, the choice which seems like it might possibly give us a convergent subsequence, is to pick some subsequence such that $\cos n_m = 1$ for all $m \in \mathbb{N}$, so let's try the most obvious such subsequence: $n_m = 2m+1$. Then

$$b_{n_m} = 2m+1 - \sqrt{(2m+1)^2+1}$$

so we only have to show that this converges. But since $\sqrt{x}$ is increasing for $x \in [1,\infty)$, we know that $\sqrt{(2m+1)^2+1} > 2m+1$ for all $m \in \mathbb{N}$, whence $b_{n_m}$ is bounded above by zero. So we now just need to show that $b_{n_m}$ is monotonically increasing. But

$$\frac{\mathrm{d}}{\mathrm{d}x}(x - \sqrt{x^2+1}) = 1 - \frac{x}{\sqrt{x^2+1}} \geqslant 0.$$

Alternatively, you could use some argument along the lines of

$$n - \sqrt{n^2+1} = n - n\sqrt{1 - \frac{1}{n^2}}$$

and $\frac{1}{n^2} \to 0$, so $1 - \frac{1}{n^2} \to 1$, so $n\sqrt{1-\frac{1}{n^2}} \to n$, so $n - n\sqrt{1-\frac{1}{n^2}} \to 0$.

**Problem 3. (2.6.D)**
*Show that every sequence has a monotone subsequence.*

Consider the subsequence that defines $\limsup$, i.e. $b_n = \sup\{a_k \mid k \leqslant n\}$.

**Problem 4. (2.6.H)**
*Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that there is a real number $L$ with the property that every subsequence $(x_{n_k})_{k=1}^{\infty}$ has a subsubsequence $\left(x_{n_{k(l)}}\right)_{l=1}^{\infty}$ with*

$$\lim_{l\to\infty} x_{n_{k(l)}} = L.$$

*Show that the whole sequence converges to $L$.*

***Hint.*** *If it were false, then you could find a subsequence bounded away from $L$.*

We will be doing this exercise in the next TD, but note two important things:

[1] This is precise enough, but just to be clear, we are saying that we need (at the very least) some $M \in \mathbb{N}$ such that $\cos n_m = -1$ for all $m > M$

(1) $(x_n)$ is a subsequence of itself; and

(2) a proof by contradiction would start with 'Assume that $(x_n)$ does *not* converge to $L$', but this can mean two *different* things — either $(x_n)$ has some limit $M$ with $L \neq M$, or $(x_n)$ has no limit at all — and so we have to find a contradiction for *both* cases individually.

**Problem 5. (2.6.J)**
*Suppose that* $(x_n)_{n=1}^{\infty}$ *is a sequence of real numbers. If* $L = \liminf x_n$, *show that there is a subsequence* $(x_{n_k})_{k=1}^{\infty}$ *so that* $\lim_{k\to\infty} x_{n_k} = L$

We can just take the subsequence that defines $\liminf$.

**Problem 6. (2.7.A)**
*Give an example of a sequence* $(a_n)$ *such that* $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$, *but the sequence does not converge.*

An easy answer is something like $a_n = \log n$, since this tends to $+\infty$. There is, however, an example that satisfies the stronger condition of 'not converging, not even to $\pm\infty$'. Consider a sequence that approximates $\sin x$, badly at first, but better as $n$ increases, i.e. a sequence $(a_n)$ such that $|a_n - \sin n| \to 0$. Then $(a_n)$ will tend to $\sin n$, which is not at all convergent.

Alternatively (as one of you pointed out), have a look at $\sin(\log n)$. This is sort of the 'perpendicular' idea to the one above, since we just take points on $\sin x$ that get closer and closer, and we know that $|\sin(x+\delta) - \sin x| \to 0$ as $\delta \to 0$, so this *looks like* it should work (see Figure 1 for a (very) rough picture).

**Problem 7. (2.7.G)**
*Evaluate the continued fraction* $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}}$.

Look at the successive terms in this sequence and make a guess as to what the $n$-th

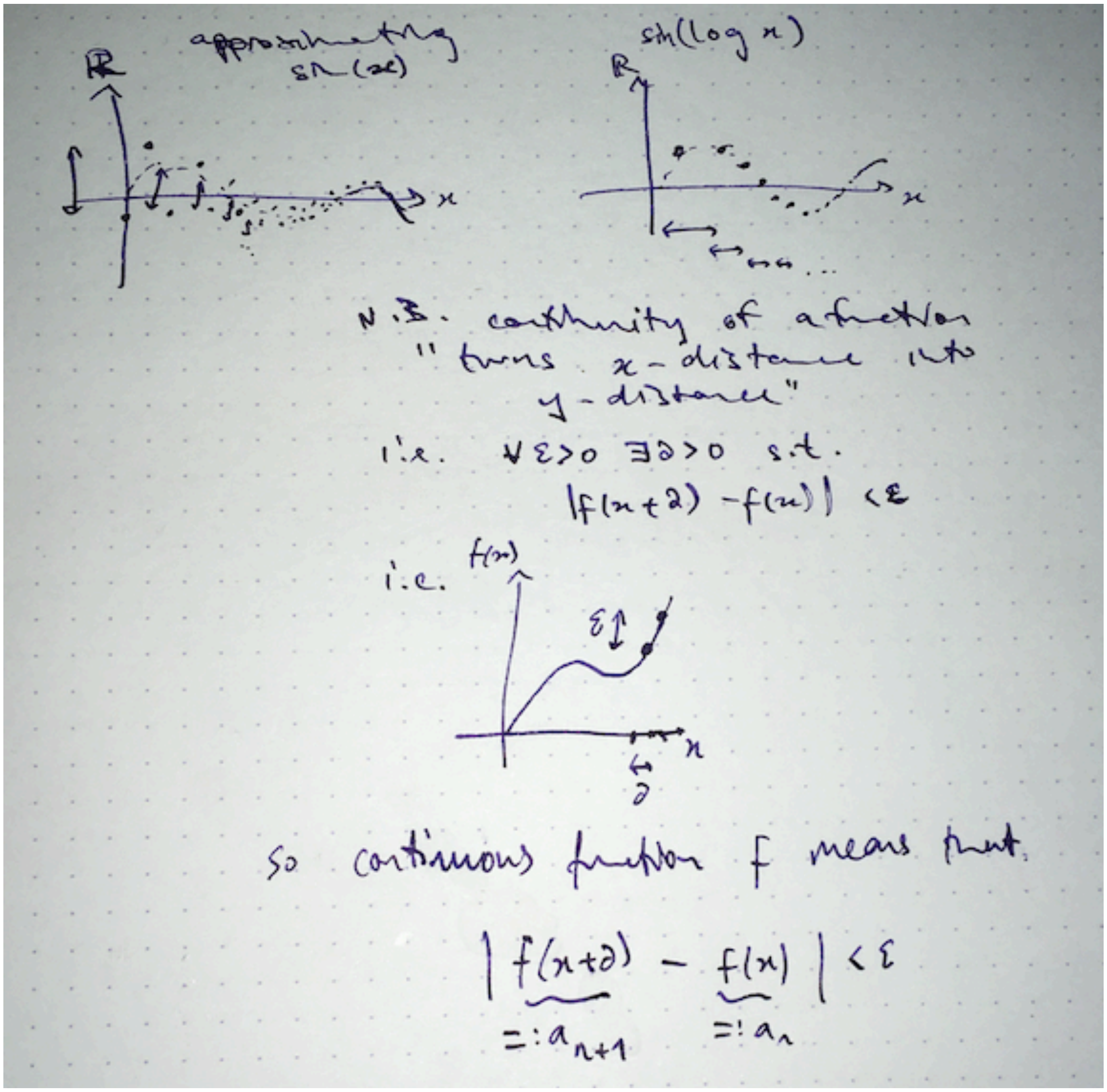


Figure 1: Problem 6.

term will be:

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 + \frac{1}{1} = 2 \\
a_2 &= 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2} \\
a_3 &= 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1}}} = \frac{5}{3} \\
a_4 &= 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}} = \frac{8}{5} \\
a_4 &= 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}} = \frac{13}{8} \\
&\dots
\end{aligned}
$$

where $F_n$ is the $n$-th Fibonacci number. Prove this (by induction). Then use the fact that $\phi \approx 1.618$ is given exactly by the limit of the ratio of successive Fibonacci numbers.

For fun, take a look at https://en.wikipedia.org/wiki/Continued_fraction#A_property_of_the_golden_ratio_%CF%86.