TD 4 solutions

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Problem 1. *Let A and B be disjoint closed subsets of $\mathbb{R}^n$. Define*

$$d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}.$$

(i) *If $A = \{a\}$ is a singleton, show that $d(A,B) > 0$.*

(ii) *If A is compact, show that $d(A,B) > 0$.*

(iii) *Find an example of A and B for $n = 2$ with $d(A,B) = 0$.*

Solution 1.

(i) In this case, $d(A,B) = \inf\{d(a,b) \mid b \in B\}$. Note that it is **not** enough to say that we are taking the infimum of a set of non-zero numbers, because there is no reason why this should be non-zero.[1] However, B is closed, and so contains all its limit points. Further, $d(a,-)\colon B \to \mathbb{R}$ is continuous, and so commutes with limits of sequences. Lastly, by the definition of the infimum, there exists some sequence $(s_n)_{n\in\mathbb{N}} \subseteq \{d(a,b) \mid b \in B\}$ such that $\lim_{n\to\infty} s_n = \inf\{d(a,b) \mid b \in B\}$, and, by definition of the set of which we are taking the infimum, there exists $(b_n)_{n\in\mathbb{N}} \subseteq B$ such that $s_n = d(a,b_n)$. Then, letting $\underline{b} = \lim_{n\to\infty} b_n$ (which exists since $\lim_{n\to\infty} s_n$ exists), we have that

$$d(A,B) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} d(a,b_n) = d(a, \lim_{n\to\infty} b_n) = d(a,\underline{b}) > 0.$$

(ii) As above, let $(s_n)_{n\in\mathbb{N}} \subseteq \{d(a,b) \mid a \in A, b \in B\}$ be such that $d(A,B) = \lim_{n\to\infty} s_n$, and let $a_n \in A, b_n \in B$ be such that $s_n = d(a_n,b_n)$. By (a more abstract statement of) Bolzano-Weierstass (combined with Heine-Borel), since A is a compact subset of $\mathbb{R}^n$, we know that there is some convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$, and we write $\underline{a} = \lim_{k\to\infty} a_{n_k} \in A$. Then, by the triangle inequality,

$$d(\underline{a}, b_{n_k}) \leqslant d(\underline{a}, a_{n_k}) + d(a_{n_k}, b_{n_k}).$$

The first term on the right-hand side tends to zero as $k \to \infty$ (since $\underline{a} = \lim_{k\to\infty} a_{n_k}$); and if $d(A,B) = 0$ then the second term would also tend to zero as $k \to \infty$, since $d(a_{n_k}, b_{n_k}) = s_{n_k}$. But this would mean that $\underline{a}$ is the limit of $(b_{n_k})_{k\in\mathbb{N}}$ as $k \to \infty$, and thus an element of B (since B is closed), which gives a contradiction (since $\underline{a} \in A$, and A and B are assumed to be disjoint), and so it cannot be the case that $d(A,B) = 0$.

(iii) Since the previous part tells us that this cannot be the case if (at least) one of A or B is compact, but the hypothesis states that both A and B are closed, we know (by Heine-Borel) that we need to look for unbounded subsets. Let $A = \{y = 0\}$ and $B = \{xy = 1, x > 0\}$, where x and y are the coordinates on $\mathbb{R}^2$. Convince yourself that this works.[2]

Problem 2. *Prove that a countable compact set $X = \{x_i \in \mathbb{R}^n \mid i \in \mathbb{N}\}$ must have isolated points.*[3]

Solution 2. This is Theorem 2.43 (p. 41) in Baby Rudin.[4]

[1] Think, for example, about the infimum of $\{\frac{1}{n} \mid n \in \mathbb{N}\}$.

[2] If you can't, then let me know!

[3] If a set is closed and has no isolated points then we say that it is *perfect*. This exercise tells us that perfect sets cannot be countable. It is also true that they cannot be finite, and so must be uncountable.

[4] That is, *Principles of Mathematical Analysis* by Walter Rudin, which can be found online, for example, at https://notendur.hi.is/vae11/%C3%9Eekking/principles_of_mathematical_analysis_walter_rudin.pdf.

Problem 3. *Show that there is a bijection between* $[0,1]$ *and the Cantor set. You can either find an explicit bijection or use the Cantor–Schröder–Bernstein Theorem.*

Solution 3. This is a classical property of the Cantor set. See, for example, `https://en.wikipedia.org/wiki/Cantor_set#Cardinality`.

Problem 4. *The Sierpinski triangle is constructed in the plane as follows. Start with a solid equilateral triangle, and call this* S_0*. Remove the open*[5] *middle triangle whose vertices are at the midpoint of each side of the larger triangle, leaving three solid equilateral triangles whose sides are half the length of the original's, and call this* S_1*. From each of these three, remove the open middle triangle just as before, leaving nine equilateral triangles whose sides are one quarter the length of the original's, and call this* S_2*. Repeat this process ad infinitum. Let* $S = \bigcap_{n\in\mathbb{N}} S_n$ *denote the intersection of all the finite stages.*

(i) *Show that* S *is a non-empty compact set.*

(ii) *Show that* S *has empty interior.*

(iii) *Show that the boundaries of the triangles at the* n*-th stage lie in* S*. Hence show that, for any* $s \in S$ *and any* $\varepsilon > 0$*, there exists a path in* S *from the top vertex of the original triangle to a point in an open ball of size* ε *centred around* s*.*

(iv) *Calculate the area that has been removed from the original triangle in order to obtain* S*.*

(v) *Construct a decision tree for* S*. Does each decision tree correspond to exactly one point in* S*? Show that* S *is uncountable.*

Figure 1: The first five stages of the construction of the Sierpinski triangle (image from Wikimedia Commons).

Solution 4. *I have yet to have the time to write up a solution for this exercise (sorry!), but you should be able to find something online after searching for 'Sierpinski triangle', because it's pretty standard.*

[5] That is, the open set given by the triangle minus its boundary.