TD 4

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Problem 1. *Let A and B be disjoint closed subsets of $\mathbb{R}^n$. Define*

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

(i) *If $A = \{a\}$ is a singleton, show that $d(A, B) > 0$.*

(ii) *If A is compact, show that $d(A, B) > 0$.*

(iii) *Find an example of A and B for $n = 2$ with $d(A, B) = 0$.*

Problem 2. *Prove that a countable compact set $X = \{x_i \in \mathbb{R}^n \mid i \in \mathbb{N}\}$ must have isolated points.*[1] *Hint: construct a family of non-empty compact sets (X_n) such that $x_n \notin X_n$ and $X_{n+1} \subseteq X_n$.*

Problem 3. *Show that there is a bijection between $[0, 1]$ and the Cantor set. You can either find an explicit bijection or use the Cantor–Schröder–Bernstein Theorem.*

Problem 4. *The Sierpinski triangle is constructed in the plane as follows. Start with a solid equilateral triangle, and call this S_0. Remove the open*[2] *middle triangle whose vertices are at the midpoint of each side of the larger triangle, leaving three solid equilateral triangles whose sides are half the length of the original's, and call this S_1. From each of these three, remove the open middle triangle just as before, leaving nine equilateral triangles whose sides are one quarter the length of the original's, and call this S_2. Repeat this process ad infinitum. Let $S = \bigcap_{n \in \mathbb{N}} S_n$ denote the intersection of all the finite stages.*

(i) *Show that S is a non-empty compact set.*

(ii) *Show that S has empty interior.*

(iii) *Show that the boundaries of the triangles at the n-th stage lie in S. Hence show that, for any $s \in S$ and any $\varepsilon > 0$, there exists a path in S from the top vertex of the original triangle to a point in an open ball of size ε centred around s.*

(iv) *Calculate the area that has been removed from the original triangle in order to obtain S.*

(v) *Construct a decision tree for S. Does each decision tree correspond to exactly one point in S? Show that S is uncountable.*

Figure 1: The first five stages of the construction of the Sierpinski triangle (image from Wikimedia Commons).

[1] If a set is closed and has no isolated points then we say that it is *perfect*. This exercise tells us that perfect sets cannot be countable. It is also true that they cannot be finite, and so must be uncountable.

[2] That is, the open set given by the triangle minus its boundary.