Take-home Exam 1

February 25, 2019

Format

This project is split into two parts: **Part A** and **Part B**.

Part A will ask you to write the equivalent of some introductory lecture notes for **one** of three subjects. This means that you should write a few pages (I could imagine anywhere between 2 and 8 sides of A4, but longer, if you wish, is allowed) where you give *the main definitions*, *the major relevant theorems* (and *proofs*, if doable), *examples*, and anything else that you think important. If you want to focus on a specific aspect of a topic and discuss things that we haven't talked about during classes then that is fine, but **not** necessary. There will be a list of things that you **must** include for each option. The goal is to produce something that any one of your classmates (who picked a different subject) could read in order to learn the subject that you picked. It may be helpful to think of this part of the project as simply *writing some very thorough revision notes for yourself, that include everything you would need to know to take an exam on the subject, and in the format you would find in a book or lecture notes*[1]. This part of the project should take no more than about **3 hours** — it is possible that you will complete this much quicker, but it definitely shouldn't take you too much longer.

Part B consists of a collection of various exercises, each with a *weight*, given in brackets at the start of the statement of the problems. You may pick any combination of the exercises that you wish as long as their weights add up to **at least ten**. For each exercise that you pick, write a full solution. When writing a solution, please make it clear for which problem it is intended. This part of the project should take no more than about **2 hours**.

Please write everything in English, and either neatly handwritten or typed (I don't mind which).

Rules

Rules are given below, but they could really be summarised as *"don't cheat"*. Breaking any rule will result in a final grade of zero.

- You will have one week to complete this project: it will be assigned on the 26th of February and is to be handed in during the TD on the 5th of March. If, for some reason, you are unable to attend the TD to hand in your project, then please email `tim.hosgood@gmail.com` to arrange some other time or place.
- You **may** use: **any** dictionary (French to English, vice versa, or just English, etc.), **be it** online[2] or physical; **any** maths book, lecture notes etc., **be it** online or physical; Wikipedia[3].
- You **may not** use: Google Translate, or **any other** such service whose use is to translate **entire sentences**[4]; web forums, to **post** questions (but reading threads that already exist is of course fine); **any** sort of translation company service, **be it** paid or free.

[1] That is, **Theorem.** Some theorem.; **Proof.** A proof. □; Some notes and motivation for the next definition.; **Definition**. A definition.; etc.

[2] I recommend WordReference.

[3] A good way of translating mathematical terms that you can't find in a dictionary is to see if the Wikipedia page is available in French and English.

[4] And even if you do this with the intent of cheating, the translations that you obtain are notoriously bad.

Mark	Description
0	The candidate either cheated or didn't submit anything.
1, 2	The project lacks a clear structure; it isn't obvious what is part of a definition and what is not; there are large mathematical mistakes; important definitions/theorems are missing; the quality of the English is generally very poor.
3, 4	The project has some vague structure; it is sometimes clear what is meant to be a definition and what is meant to be a theorem; there are a few big mathematical mistakes; important definitions/theorems are missing; the quality of the English is generally poor.
5, 6	The project clearly has narrative structure; it is generally clear what is meant to be a definition and what is meant to be a theorem; there are a few mathematical errors, but none too severe; most of the expected content is included; the quality of the English is generally good.
7, 8	The project has a good structure, and there are some motivating explanations given between the statements of theorems and definitions etc.; the formatting is careful and clear, with clear definitions, theorems, and proofs; there are only a few mathematical errors, and all minor; all of the expected content is included; the quality of the English is very good.
9, 10	The project has an excellent structure, with choice examples given to motivate key definitions and theorems; there is no confusion in understanding what is part of a definition and what is part of a theorem; there are basically no mathematical errors; there is content included which clearly shows that the candidate engaged in additional research; the quality of the English is excellent.

Table 1: The mark scheme for Part A.

- You **may not** allow other students to proof-read your project, or to write any part of it for you, **nor** may you proof-read the project of another student, or write any part of another student's project.
- You **may not** copy verbatim from **any** other source, **unless** properly cited.
- You **may** discuss the *logistics* of your project with other students, i.e. "I think I'm going to pick option 1 for Part A", "I've written about 5 pages — how much have you written for this part?", "I think that problem (vi) is too hard to be worth doing". *I understand that this rule seems a bit vague, but I do* ***not*** *want to forbid you from talking to other students entirely. Please use common sense when debating what counts as cheating and what doesn't — I think the spirit of these rules is really rather clear: the work that you are submitting should be entirely your own.*

Marking

Part A will be marked out of **ten**, and **Part B** will be marked out of **twenty**, and then a weighted average of the two marks will be calculated, with **Part A** accounting for about 65% of the final grade.

The criteria by which I will mark **Part A** is given in Table 1. For **Part B**, each question will be given a mark out of *twice its weight* (e.g. a question of weight 2 will be marked out of a maximum of 4 points), and the marking will be the same as in a 'normal' exam (i.e. full marks means that the solution is clear, well written, and at least almost entirely correct) but the main aim is to write **clear** and **understandable** formal proofs. Keep in mind that a good proof should do more than just 'be true': it should help the reader to understand *why* it is true.

Part A

Pick any **one** of the following. The things listed after the name are things that you *must* include, **except** for things listed under '*Other*', which are simply ideas for further content, if you so wish. You may include as much additional content as you wish, but this is *not* necessary. You may use any of the exercises from any of the TD sheets as examples, or any that you can invent yourself, find in a book, on TD sheets for other classes, etc. You may **not**, however, blindly copy solutions given in whichever book, TD sheet, etc., you find the example in.

1. **Formal proofs.** *Definitions:* direct proof; proof by contradiction; proof by induction; elementary logic (i.e. the truth tables of `AND` and `OR`). *Theorems:* de Morgan's laws: $X \setminus (P \cup Q) = (X \setminus P) \cap (X \setminus Q)$ and $X \setminus (P \cap Q) = (X \setminus P) \cup (X \setminus Q)$, for any set X and subsets $P, Q \subseteq X$. *Examples:* at least two examples of each type of proof, with at least one of the examples for a proof by induction being induction on something other than $\mathbb{N}$ (e.g. on odd numbers, multiples of 3, etc.); a constructive and a non-constructive proof (with an explanation of the difference between the two). *Other:* when can we apply a proof by induction? e.g. can we do induction over $\mathbb{R}$? over the set of prime numbers?; are there any other 'main' types of proofs?; do constructive proofs always exist?.

2. **Dedekind cuts.** *Definitions:* Dedekind cuts; real numbers (constructed via Dedekind cuts) — *from this point on you may assume that $\mathbb{R}$ constructed via Dedekind cuts has all the properties and operations that we already know that $\mathbb{R}$ has, but you **must** make it clear what assumptions you make, e.g. that addition of Dedekind cuts corresponds to addition of real numbers* — supremum and infimum (of a subset of $\mathbb{R}$); maximum and minimum (of a subset of $\mathbb{R}$); $\liminf$ and $\limsup$. *Theorems:* a Dedekind cut corresponds to a rational number if and only if its complement in $\mathbb{R}$ is *not* a Dedekind cut; bounded above implies existence of the supremum; the limit of a sequence exists if and only if the $\liminf$ and $\limsup$ are equal, and then it is equal to both of them. *Examples:* the Dedekind cut corresponding to some specific irrational number that is **not** π or $\sqrt{n}$ for some $n \in \mathbb{Z}$; sets that have maximums (or minimums) and sets that don't. *Other:* what other ways are there of defining $\mathbb{R}$, and how do they compare to Dedekind cuts? are there good reasons to pick a different method over Dedekind cuts?

3. **Cauchy sequences.** *Definitions:* Cauchy sequences; Cauchy completeness; partial sums of a series $\sum_{n=0}^{\infty} a_n$; convergence of a series $\sum_{n=0}^{\infty} a_n$. *Theorems:* Bolzano-Weierstrass; every convergent sequence is Cauchy; every Cauchy sequence is bounded; a sequence of real numbers is convergent if and only if it is Cauchy. *Examples:* a sequence (a_n) such that $|a_{n+1} - a_n| \to 0$ but that is *not* Cauchy; a sequence of *rational* numbers that is Cauchy but does *not* converge in $\mathbb{Q}$. *Other:* examples of 'nasty' sequences and series.

As a rough guide: option 1 (formal proofs) is probably the shortest of the three, and so maybe(!) the easiest with which to get an average mark; the other two options are probably longer, but therefore maybe(!) easier to elaborate upon, if you wish to aim for a higher mark.

Part B

Pick *any number* of the following exercises, such that their weights add up to *at least ten*. Weights are given in brackets at the start of each problem.

(I) (5) Let $X = \{0, 1, 2\}$ and $Y = \{0, 1, 2, 3, 4\}$. How many functions from X to Y are there? How many functions from Y to X? How many injective functions from X to Y are there? How many injective functions from Y to X? How many surjective functions from X to Y are there? How many surjective functions from Y to X?

Note: this problem has a high weight not because it is necessarily hard, but because a good proof involves more than just listing all the functions — you should explain why *your list is exhaustive, and ideally present it in such a way that it is clear how to generalise this method for larger sets.*

(II) (4) Describe all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy $f(x) = f\big((1+x)/2\big)$.

(III) (3) The following is an *invalid* proof of the AM-GM inequality (the fact that $\sqrt{xy} \leqslant \frac{x+y}{2}$ for all non-negative real numbers x, y, with equality if and only if $x = y$).

> We know that $x, y \geqslant 0$. Suppose that $\sqrt{xy} \leqslant \frac{x+y}{2}$, so that $2\sqrt{xy} \leqslant x+y$. Then $4xy \leqslant (x+y)^2$, which is the same as saying that $0 \leqslant x^2 - 2xy + y^2 = (x-y)^2$. But the square of any real number is positive, and so this is clearly true, and we are done.

Explain why this proof is invalid, and give a correct version of the proof.

(IV) (3) Let X, Y be finite sets such that $X \cap Y = \varnothing$. Prove that $X \cup Y$ is finite, and that $|X \cup Y| = |X| + |Y|$, where $|\cdot|$ denotes the cardinality (i.e. number of elements) of the set. *Hint: use induction.*

(V) (3) Define φ to be the larger positive root of the polynomial $x^2 - x - 1$. Prove that

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$$

as well as formalising what "…" means here.

(VI) (2) Does the sequence $a_n = \cos \log n$ converge as $n \to \infty$? Give a proof of your answer.

(VII) (2) Prove that, for any natural number n, both n and its digit sum[5] have the same remainder when divided by 9.

(VIII) (2) Let $f\colon X \to Y$ be a function between sets, and $A, B \subseteq Y$ two subsets. Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, where f^{-1} denotes the preimage, i.e. $f^{-1}(S) := \{x \in X \mid f(x) \in S\}$.

(IX) (1) Let X, Y, Z be sets. Prove that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

(X) (1) Prove that $x \sim y \iff f(x) = f(y)$ gives an equivalence relation on X for any function of sets $f\colon X \to Y$.

[5] That is, the sum of the digits (in its base-10 representation). So the digit sum of 1283 is $1+2+8+3 = 14$, the digit sum of 1000012 is 4, and the digit sum of 6 is 6.